Test 2 **MA 464 (Theory of Probability)** **November 16, 2017**

Name: ______________________ **Signature:** ______________________

SHOW ALL YOUR WORK!

1. [20 points] Let X be a continuous random variable with the distribution function

$$F_X(x) = \begin{cases} \frac{1}{80}(x^4 - 1) & 1 \leq x \leq 3 \\ 1 & x > 3 \\ 0 & x < 1 \end{cases}.$$

   (a) Find the density function of X, $f_X(x)$.

   (b) Find a formula for the $k-$th moments of X, i.e., $\mathbb{E}(X^k)$ for $k \geq 1$.

2. [15 points] **Solve ONE of the following two questions**

Q 1 Let

$$F(x) = \int_{-\infty}^{x} c\, e^{-|u|}\, du.$$

Find the value of the constant c that defines F as a distribution function.

Q 2 Let X be an exponential random variable with parameter $\lambda > 0$. Find the distribution function and the density function for the random variable $Y = 1 - \sqrt{X}$.

3. [20 points] Suppose X and Y are independent standard normal random variables. Find the distribution of $Z = \sqrt{X^2 + Y^2}$.

4. [15 points] Let X and Y be two independent random variables from Exp(1).

   (a) Find the joint density function of X and Y, $f_{X,Y}(x, y)$.

   (b) Find $\mathbb{P}(X + Y \leq 1)$.

5. [20 points] Let $X_1, \ldots, X_n$ be i.i.d random variables from $U(0,1)$. Denote $V = \max\{X_1, \ldots, X_n\}$ and $W = \min\{X_1, \ldots, X_n\}$.

(a) Find the distribution functions and the density functions of each of V and W.

(b) Find $\mathbb{E}(\frac{V+W}{2})$.

6. [15 points] Let $X \sim N(0,1)$ and $Y \sim \text{Exp}(2)$ be independent random variables, and let $Z_1 = X+Y$ and $Z_2 = 2Z_1+1$

(a) Find the moment generating function of Z_1, i.e., $M_{Z_1}(t)$.

(b) Find $M_{Z_2}(t)$.

(c) Find the correlation $\rho(Z_1, Z_2)$.

1- Discrete Random variables (for $0 \le p \le 1$, we use $q := 1 - p$)

Bernoulli: $X \sim \text{Bernoulli}(p)$

$$\mathbb{P}(X=1) = p \quad \text{and} \quad P(X=0) = q\,.$$
$$\mathbb{E}(X) = p, \quad \text{Var}(X) = pq, \;\; M_X(t) = q + pe^t.$$

Binomial: $X \sim \text{Binomial}(n,p)$, $n \in \mathbb{N}$.

$$\mathbb{P}(X=k) = \binom{n}{k} q^{n-k} p^k, \quad k = 0, 1, \ldots, n.$$
$$\mathbb{E}(X) = np, \quad \text{Var}(X) = npq, \;\; M_X(t) = (q + pe^t)^n.$$

Geometric: $X \sim \text{Geometric}(p)$

$$\mathbb{P}(X=k) = q^{k-1}p, \quad k = 1, 2, \ldots$$
$$\mathbb{E}(X) = \frac{1}{p}, \quad \text{Var}(X) = \frac{q}{p^2}, \;\; M_X(t) = \frac{pe^t}{1-qe^t}.$$

Poisson: $X \sim \text{Poisson}(\lambda)$, $\lambda > 0$.

$$\mathbb{P}(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, \ldots$$
$$\mathbb{E}(X) = \lambda, \quad \text{Var}(X) = \lambda, \;\; M_X(t) = e^{\lambda(e^t-1)}.$$

Negative Binomial: $X \sim \text{NB}(n,p)$, $n \in \mathbb{N}$.

$$\mathbb{P}(X=k) = \binom{k-1}{n-1} p^n q^{k-n}, \quad k = n, n+1, \ldots$$
$$\mathbb{E}(X) = \frac{n}{p}, \quad \text{Var}(X) = \frac{nq}{p^2}, \;\; M_X(t) = \left(\frac{pe^t}{1-qe^t}\right)^n.$$

2- Continuous Random variables

Uniform: $X \sim U(a,b)$, where $a < b$.

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0 & \text{otherwise} \end{cases}$$
$$\mathbb{E}(X) = \frac{a+b}{2}, \quad \text{Var}(X) = \frac{(b-a)^2}{12}, \quad M_X(t) = \frac{e^{bt}-e^{at}}{t(b-a)}, \; t \neq 0.$$

Exponential: $X \sim \text{Exp}(\lambda)$, where $\lambda > 0$.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$
$$\mathbb{E}(X) = \frac{1}{\lambda}, \quad \text{Var}(X) = \frac{1}{\lambda^2}, \quad M_X(t) = \frac{\lambda}{\lambda - t}, \; t < \lambda.$$

Normal: $X \sim N(\mu, \sigma^2)$, where $\mu, \sigma \in \mathbb{R}$.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad -\infty < x < \infty.$$
$$\mathbb{E}(X) = \mu, \quad \text{Var}(X) = \sigma^2, \quad M_X(t) = e^{\mu t + \frac{(\sigma t)^2}{2}}.$$

Cauchy:

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$
$$\mathbb{E}(X) = \text{undefined}, \quad \text{Var}(X) = \text{undefined}.$$

Gamma: $X \sim \Gamma(\omega, \lambda)$, where $\omega, \lambda > 0$.

$$f_X(x) = \begin{cases} \frac{1}{\Gamma(\omega)} \lambda^\omega x^{\omega-1} e^{-\lambda x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}, \quad \Gamma(\omega) = \int_0^\infty x^{\omega-1} e^{-x} dx.$$
$$\mathbb{E}(X) = \frac{\omega}{\lambda}, \quad \text{Var}(X) = \frac{\omega}{\lambda^2}, \quad M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^\omega, \; t < \lambda.$$